

Uniqueness of a phaseless inverse scattering problem for the generalized 3-D Helmholtz equation *

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Abstract

An inverse scattering problems for the 3-D generalized Helmholtz equation is considered. Only the modulus of the complex valued scattered wave field is assumed to be measured and the phase is not measured. Uniqueness theorem is proved.

Keywords: phaseless inverse scattering, generalized Helmholtz equation, uniqueness

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1 Introduction

Phaseless Inverse Scattering Problems (PISPs) arise in imaging of nanostructures [10, 11, 35, 41] and biological cells [36, 37]. In these applications, sizes of objects of interest are typically on the micron scale or less. Recall that $1\text{micron} = 1\mu m = 10^{-6}m$ where “ m ” stands for meter. Sizes of many nanostructures are usually hundreds of nanometers (nm), $100nm = 10^{-7}m = 0.1\mu m$. Sizes of biological cells are in the range of $(5, 100)\mu m$ [36, 37]. Therefore, in imaging of these objects the wavelengths of the electromagnetic signals should also be in the same range. As to those signals, usually either X-rays or optical signals are used. It is well known that the phase of an electromagnetic signal cannot be measured for such small wavelengths [10, 11, 35, 41]. Therefore, we arrive at the problem of the reconstruction of the spatially distributed dielectric constant of a scatterer

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using only the intensity of the scattered wave field. The intensity is the square modulus of the complex valued wave field.

A similar problem, although for the Schrödinger equation in the frequency domain, arises in the quantum inverse scattering, where only the differential scattering cross-section can be measured, i.e. the square modulus of the solution of that equation, see, e.g. page 8 of [30] and Chapter 10 of [8]. Unlike the generalized Helmholtz equation considered in this paper, in the case of the Schrödinger equation the unknown potential is not multiplied by k^2 , which significantly simplifies the problem. Note that, unlike PISPs, the conventional inverse scattering theory is based on the assumption that both the intensity and the phase of the complex valued wave field are measured, see, e.g. [4, 8, 9, 13, 14, 29, 30, 31, 32]. Here and below k is the wave number.

Let $\Omega, G \subset \mathbb{R}^3$ be two bounded domains and $\Omega \subset G$. Let $S = \partial G, S \in C^1$ and $S \cap \partial\Omega = \emptyset$. Let $\omega > 0$ be a number. For every $y \in \mathbb{R}^3$ denote $B_\omega(y) = \{x \in \mathbb{R}^3 : |x - y| < \omega\}$ the ball of the radius ω with the center at the point y . Below $c(x), x \in \mathbb{R}^3$ is the spatially distributed dielectric constant. We assume that $c(x)$ is a real valued function satisfying the following conditions

$$c \in C^{15}(\mathbb{R}^3), \quad c(x) = 1 + \beta(x), \quad (1)$$

$$\beta(x) \geq 0, \quad \beta(x) = 0 \quad \text{for } x \in \mathbb{R}^3 \setminus \Omega. \quad (2)$$

The smoothness requirement imposed on the function $c(x)$ is due to Lemma 2.1 (below) as well as due to Theorem 2 in [22]. The conformal Riemannian metric generated by the function $c(x)$ is

$$d\tau = \sqrt{c(x)} |dx|, \quad |dx| = \sqrt{(dx_1)^2 + (dx_2)^2 + (dx_3)^2}. \quad (3)$$

Below we rely on the following Assumption:

Assumption. *We assume that geodesic lines of the metric (3) satisfy the regularity condition, i.e. for each two points $x, y \in \mathbb{R}^3$ there exists a single geodesic line $\Gamma(x, y)$ connecting these points.*

It is well known from the Hadamard-Cartan theorem [5] that in any simply connected complete manifold with a non positive curvature each two points can be connected by a single geodesic line. The manifold (Ω, ε_r) is called the manifold of a non positive curvature, if the section curvatures $K(x, \sigma) \leq 0$ for all $x \in \overline{\Omega}$ and for all two-dimensional planes σ . A sufficient condition for the inequality $K(x, \sigma) \leq 0$ was derived in [40]. This condition is:

$$\sum_{i,j=1}^3 \frac{\partial^2 \ln c(x)}{\partial x_i \partial x_j} \xi_i \xi_j \geq 0, \quad \forall \xi \in \mathbb{R}^3, \forall x \in \overline{\Omega}. \quad (4)$$

Thus, (4) is a sufficient condition, which guarantees the validity of Assumption in terms of the function $c(x)$. For $x, y \in \mathbb{R}^3$, let $\tau(x, y)$ be the solution to the following problem:

$$|\nabla_x \tau(x, y)|^2 = c(x), \quad (5)$$

$$\tau(x, y) = O(|x - y|) \quad \text{as } x \rightarrow y. \quad (6)$$

Equation (5) is called “eikonal equation”. Let $d\sigma$ be the euclidean arc length of the geodesic line $\Gamma(x, y)$. Then the solution of the problem (5), (6) is

$$\tau(x, y) = \int_{\Gamma(x, y)} c(\xi) d\sigma. \quad (7)$$

Hence, $\tau(x, y)$ is the travel time between points x and y due to the Riemannian metric (3). Due to the Assumption, $\tau(x, y)$ is a single-valued function of both points x and y in $\mathbb{R}^3 \times \mathbb{R}^3$.

We consider the following equation

$$\Delta u + k^2 c(x)u = -\delta(x - y), \quad x \in \mathbb{R}^3, \quad (8)$$

where the Laplace operator is taken with respect to x , the wave number $k > 0$ and $y \in \mathbb{R}^3$ is the source position. Naturally, we assume that the function $u(x, y, k)$ satisfies the radiation condition

$$\frac{\partial u}{\partial r} - iku = o(r^{-1}) \quad \text{as } r = |x - y| \rightarrow \infty. \quad (9)$$

Denote $u_0(x, y, k)$ the solution of the problem (8), (9) for the case $n(x) \equiv 1$. Then u_0 is the incident spherical wave,

$$u_0(x, y, k) = \frac{\exp(ik|x - y|)}{4\pi|x - y|}. \quad (10)$$

Let $u_{sc}(x, y, k)$ be the scattered wave, which is due to the presence of scatterers, in which $c(x) \neq 1$. Then

$$u_{sc}(x, y, k) = u(x, y, k) - u_0(x, y, k) = u(x, y, k) - \frac{\exp(ik|x - y|)}{4\pi|x - y|}. \quad (11)$$

Combining Theorem 8.7 of [9] with Theorem 6.17 of [12] and taking into account that $c \in C^{15}(\mathbb{R}^3)$, we obtain that the problem (8), (9) has unique solution

$u \in C^{16+\alpha}(|x - x_0| \geq \varepsilon), \forall \varepsilon > 0$ for any $\alpha \in (0, 1)$. Here $C^{16+\alpha}$ is the Hölder space.

We model the propagation of the electric field by a single equation (8) since it was demonstrated numerically in [7] that this is possible in the case when only one component of this field is incident upon the medium. Indeed, it was shown in [7] that in this case that component dominates two other components and also its propagation is well governed by a single PDE, which is a direct analog of (8). This conclusion was verified in [6, 42, 43] via accurate imaging from experimental data.

Denote $\text{dist}(S, \partial\Omega) > 0$ the Hausdorf distance between the surface S and the domain Ω .

Phaseless Inverse Scattering Problem (PISP). Let the number $\omega \in (0, \text{dist}(S, \partial\Omega))$. Let $u(x, y, k)$ be the solution of the problem (8), (9). Assume that the following function $f(x, y, k)$ is known

$$f(x, y, k) = |u(x, y, k)|, \forall y \in S, \forall x \in B_\omega(y), x \neq y, \forall k \in (a, b), \quad (12)$$

where $(a, b) \subset \{z > 0\}$ is a certain interval. Determine the function $c(x)$.

Theorem 1 is the main results of this paper. Compared with previous results [19, 20, 21], the main difficulty in the proof of this theorem is due to the necessity of using the apparatus of the Riemannian geometry.

Theorem 1. *Consider an arbitrary pair of points $y \in S, x \in B_\omega(y), x \neq y$. And consider the function $g_{x,y}(k) = f(x, y, k)$ as the function of the variable k , where the function $f(x, y, k)$ is defined in (12). Then the function $\varphi_{x,y}(k) = u(x, y, k)$ of the variable k is reconstructed uniquely, as soon as the function $g_{x,y}(k)$ is given for all $k \in (a, b)$. The PISP has at most one solution.*

The first uniqueness theorem for a PISP for a 1-D Schrödinger equation was proved in [17]. More recently uniqueness theorems were proved for 3-D PISPs for the Schrödinger equation in [19, 20]. Also, in [21] uniqueness was proved for the 3-D PISP for the generalized Helmholtz equation in the case when the function $\delta(x - y)$ in the right hand side of (8) was replaced by a function $p(x)$ such that

$$p(x) \neq 0 \text{ in } \overline{\Omega}. \quad (13)$$

Our PISP is overdetermined. Indeed, the unknown coefficient $c(x)$ depends on three variables, whereas the data f depend on six variables. On the other hand, even if the phase is known, currently there are no uniqueness theorems, which would be proven for a non-overdetermined statement of a 3-D inverse scattering problem in the case when the δ -function is in the

right hand side of (8). On the other hand the PISP in [21] is non over-determined. In fact, that is an inverse problem with the data generated by a single measurement event. However, the price to pay in [21], is the assumption (13). The same is true for the PISP for the Schrödinger equation on page 397 of [19].

Reconstruction procedures for PISPs both for the generalized Helmholtz equation and for the Schrödinger equation were developed in [22, 23, 24, 25]. In [26] an essentially modified procedure of [22] was implemented numerically. In [33, 34] different statements of PISPs were proposed, which led to different uniqueness theorems and reconstruction procedures. While we are interested in the reconstruction of the unknown coefficient of a PDE, we also refer to [1, 2, 3, 15, 16] for numerical solutions of PISPs in the case when the surface of a scatterer was reconstructed.

Everywhere below we assume that conditions of Theorem 1 as well as Assumption hold. In section 2 we prove some lemmata. In section 3 we prove Theorem 1.

2 Lemmata

For any complex number $z \in \mathbb{C}$ we denote by \bar{z} its complex conjugate. Let $\gamma > 0$ be a number. Denote

$$\mathbb{C}_\gamma = \{z \in \mathbb{C} : \operatorname{Im} z > -\gamma\}, \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

Hence, \mathbb{C}_+ is the upper half plane of the complex plane \mathbb{C} .

2.1 Some properties of the function $u(x, y, k)$

Let $\zeta = (\zeta_1, \zeta_2, \zeta_3)$, $\zeta = \zeta(x, y)$ be geodesic coordinates of a variable point x with respect to a fixed point y in the above Riemannian metric (3). By the Assumption, there exists a one-to-one correspondence $x \Leftrightarrow \zeta$ for any fixed y . Consider the Jacobian $J(x, y)$,

$$J(x, y) = \det \frac{\partial \zeta}{\partial x}. \quad (14)$$

It was proven in [22] that $J(x, y) > 0$ for all x, y . Let $T > 0$ be an arbitrary number. Consider the domains $D(y, T)$ and $D^*(y, T)$ defined as

$$\begin{aligned} D(y, T) &= \{(x, t) : 0 < t \leq T - \tau(x, y)\}, \\ D^*(y, T) &= \{(x, t) : \tau(x, y) \leq t \leq T - \tau(x, y)\}. \end{aligned}$$

Lemma 2.1. *Let $y \in \mathbb{R}^3$ be an arbitrary point. Then there exists a number $\gamma = \gamma(y, G) > 0$ such that for every $x \in G$ the function $u(x, y, k)$ can be analytically continued with respect to k from the half real line $\mathbb{R}_+ = \{k : k > 0\}$ in the half plane \mathbb{C}_γ .*

Proof. Consider the following associated Cauchy problem

$$c(x) v_{tt} = \Delta v + \delta(x - y, t), x \in \mathbb{R}^3, t > 0, \quad (15)$$

$$v|_{t < 0} \equiv 0. \quad (16)$$

It was proven in [22] that the solution of this problem can be represented as

$$v(x, y, t) = A(x, y)\delta(t - \tau(x, y)) + \hat{v}(x, y, t)H(t - \tau(x, y)), \quad (17)$$

where the function $\hat{v}(x, y, t) \in C^2(D^*(y, T))$,

$$A(x, y) = \frac{\sqrt{J(x, y)}}{4\pi n(x)\tau(x, y)} \quad (18)$$

and $H(t)$ is the Heaviside function,

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}$$

It follows from Lemma 6 of Chapter 10 of the book [45] as well as Remark 3 after that lemma that for any fixed point $y \in \mathbb{R}^3$ the function $v(x, y, t)$ decays exponentially as $t \rightarrow \infty$ together with its x -derivatives up to the second order. This decay is uniform for all $x \in \overline{G}$. In other words, there exist constants $M = M(G, c) > 0, m = m(G, c) > 0$ such that

$$|D_{x,t}^\alpha v(x, y, t)| \leq M e^{-mt}, \forall t \geq t_0, \forall x \in \overline{G}, \quad (19)$$

where $t_0 = t_0(G, c) = \text{const.} > 0$. Here $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the multiindex with non-negative integer coordinates and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 2$. Hence, one can consider the Fourier transform $W(x, y, k)$ of the function v ,

$$W(x, y, k) = \int_0^\infty v(x, y, t) \exp(ikt) dt. \quad (20)$$

Next, theorem 3.3 of [44] and theorem 6 of Chapter 9 of [45] guarantee that

$$W(x, y, k) = u(x, y, k), \quad (21)$$

where the function $u(x, y, k)$ is the above solution of the problem (8), (9).

Hence, it follows from (19) and (20) that the function $u(x, y, k)$ can be analytically continued from \mathbb{R}_+ in the half plane \mathbb{C}_m . \square

An analog of Lemma 2.2 was proven in [22] for the case when k is a real number.

Lemma 2.2. *Let $A(x, y)$ be the function defined in (18). The asymptotic behavior of the function $u(x, y, k)$ is*

$$u(x, y, k) = A(x, y)e^{ik\tau(x, y)} \left(1 + O\left(\frac{1}{k}\right) \right), |k| \rightarrow \infty, k \in \mathbb{C}_m, x \in \overline{G}. \quad (22)$$

Proof. By (17) and (20)

$$u(x, y, k) = A(x, y)e^{ik\tau(x, y)} + \int_{\tau(x, y)}^{\infty} \hat{v}(x, y, t) \exp(ikt) dt. \quad (23)$$

The integration by parts in (23) and (19) lead to (22). \square

2.2 Four more lemmata

Lemma 2.3. *For any pair $x \in \overline{G}, y \in \mathbb{R}^3, x \neq y$ the function $u(x, y, k)$ has at most finite number of zeros in \mathbb{C}_m .*

Proof. Follows immediately from (22). \square

Lemma 2.4. *Let the function $r(k)$ be analytic in the half plane \mathbb{C}_m and has no zeros in $\mathbb{C}_+ \cup \mathbb{R}$. Assume that*

$$r(k) = \frac{C}{k^n} [1 + o(1)] \exp(ikL), |k| \rightarrow \infty, k \in \mathbb{C}_+,$$

where $C \in \mathbb{C}, n, L \in \mathbb{R}$ are some numbers and also $n \geq 0$. Then the function $r(k)$ can be uniquely determined for $k \in \mathbb{C}_+ \cup \mathbb{R}$ by the values of $|r(k)|$ for $k \in \mathbb{R}$.

Lemma 2.4 follows immediately from Proposition 4.2 of [18]. Hence, we omit the proof.

Lemma 2.5. *Let the function $r(k)$ be analytic for all $k \in \mathbb{R}$. Then the function $|r(k)|$ can be uniquely determined for all $k \in \mathbb{R}$ by the values of $|r(k)|$ for $k \in (a, b)$.*

Proof. We have $|r(k)|^2 = r(k) \bar{r}(k)$. Both $r(k)$ and $\bar{r}(k)$ are analytic functions of the real variable k . \square

Lemma 2.6. *Consider two finite sets of non-negative integers $\{p_{j_1}\}_{j_1=1}^{N_1}$ and $\{q_{j_2}\}_{j_2=1}^{N_2}$. Also, consider two sets of complex numbers $\{d_{j_1}\}_{j_1=1}^{N_1} \subset$*

$(\overline{\mathbb{C}}_+ \setminus \mathbb{R})$ and $\{s_{j_2}\}_{j_2=1}^{N_2} \subset (\overline{\mathbb{C}}_+ \setminus \mathbb{R})$. Assume that there exist two sets of complex numbers $\{\alpha_{j_1}\}_{j_1=1}^{N_1}$ and $\{\beta_{j_2}\}_{j_2=1}^{N_2}$ such that

$$\sum_{j_1=1}^{N_1} \alpha_{j_1} t^{p_{j_1}} \exp(-i\bar{d}_{j_1} t) = \sum_{j_2=1}^{N_2} \beta_{j_2} t^{q_{j_2}} \exp(-i\bar{s}_{j_2} t), \forall t \geq 0. \quad (24)$$

Then $N_1 = N_2 = N$ and numbers involved in (24) can be re-numbered in such a way that

$$\alpha_j = \beta_j, p_j = q_j, d_j = s_j, \forall j = 1, \dots, N.$$

Proof. Let $n \geq 0$ be an integer and let the number $a \in \overline{\mathbb{C}}_+ \setminus \mathbb{R}$. Consider the function $f_{n,a}(t)$,

$$f_{n,a}(t) = H(t) t^n \exp(-i\bar{a}t). \quad (25)$$

Let $F_{n,a}(k)$ be the Fourier transform of $f_{n,a}(t)$,

$$F_{n,a}(k) = \mathcal{F}(f_{n,a})(k) = \int_{-\infty}^{\infty} f_{n,a}(t) \exp(ikt) dt. \quad (26)$$

Then the elementary calculation shows that

$$F_{n,a}(k) = \frac{(-1)^n n!}{i^{n+1}} \cdot \frac{1}{(k - \bar{a})^{n+1}}. \quad (27)$$

Hence, multiplying both sides of (24) by the Heaviside function $H(t)$ and applying then the operator \mathcal{F} to both sides of the resulting equality, we obtain for all $k \in \mathbb{R}$

$$\sum_{j_1=1}^{N_1} \alpha_{j_1} \frac{(-1)^{p_{j_1}} (p_{j_1})!}{i^{p_{j_1}+1}} \cdot \frac{1}{(k - \bar{d}_{j_1})^{p_{j_1}+1}} = \sum_{j_2=1}^{N_2} \beta_{j_2} \frac{(-1)^{q_{j_2}} (q_{j_2})!}{i^{q_{j_2}+1}} \cdot \frac{1}{(k - \bar{s}_{j_2})^{q_{j_2}+1}}. \quad (28)$$

Since (28) is valid for all $k \in \mathbb{R}$ we can analytically continue both sides of (28) in \mathbb{C} and obtain then meromorphic functions in both sides of (28). In other words, (28) is valid for all $k \in \mathbb{C}$, except of poles. The rest of the proof is obvious. \square

3 Proof of Theorem 1

Consider two arbitrary points $y \in S, x \in B_\omega(y), x \neq y$ mentioned in the formulation of this theorem. Denote $\psi(k) = u(x, y, k)$. Suppose that there exist two functions

$$\psi_1(k) = u_1(x, y, k), \psi_2(k) = u_2(x, y, k) \quad (29)$$

such that $|\psi_1(k)| = |\psi_2(k)| = f(x, y, k), k \in (a, b)$. Then by Lemma 2.5

$$|\psi_1(k)| = |\psi_2(k)| = f(x, y, k), \forall k \in \mathbb{R}. \quad (30)$$

By Lemma 2.1 functions $\psi_1(k)$ and $\psi_2(k)$ can be analytically continued in the half plane $\mathbb{C}_m \subset \mathbb{C}$. Also, recall that the upper half plane $\mathbb{C}_+ \subset \mathbb{C}_m$.

Below we count each zero of any of two functions $\psi_1(k)$ and $\psi_2(k)$ as many times as its multiplicity is. First, we show that real zeros of functions $\psi_1(k)$ and $\psi_2(k)$ coincide. Let $\{\alpha_j^{(1)}\}_{j=1}^{r_1}$ and $\{\alpha_s^{(2)}\}_{s=1}^{r_2}$ be all real zeros of the function $\psi_1(k)$ and $\psi_2(k)$ respectively. Let $\alpha_1^{(1)}$ be the real zero of $\psi_1(k)$ of the multiplicity $n \geq 1$. Also let $\alpha_1^{(1)}$ be the zero of the function $\psi_2(k)$ of the multiplicity $m \geq 0$. Then

$$\psi_1(k) = (k - \alpha_1^{(1)})^n \widehat{\psi}_1(k), \psi_2(k) = (k - \alpha_1^{(1)})^m \widehat{\psi}_2(k), \quad (31)$$

$$\widehat{\psi}_1(\alpha_1^{(1)}) \neq 0, \widehat{\psi}_2(\alpha_1^{(1)}) \neq 0. \quad (32)$$

Hence, by (30) and (31)

$$|k - \alpha_1^{(1)}|^n |\widehat{\psi}_1(k)| = |k - \alpha_1^{(1)}|^m |\widehat{\psi}_2(k)|, \forall k \in \mathbb{R}. \quad (33)$$

Let, for example $n > m$. Dividing both sides of (33) by $|k - \alpha_1^{(1)}|^m$, we obtain

$$|\widehat{\psi}_2(k)| = |k - \alpha_1^{(1)}|^{n-m} |\widehat{\psi}_1(k)|, \forall k \in \mathbb{R}.$$

Hence, $\widehat{\psi}_2(\alpha_1^{(1)}) = 0$, which, however, contradicts to (32). Hence, we have proven that real zeros of both functions $\psi_1(k)$ and $\psi_2(k)$ coincide. Let the set of real zeros of each of these functions be $\{\alpha_j\}_{j=1}^r$.

Consider now complex zeros of functions $\psi_1(k)$ and $\psi_2(k)$ in the upper half plane \mathbb{C}_+ . By Lemma 2.3 each of these two functions has at most a finite number of zeros in \mathbb{C}_+ . Let $\{a_j\}_{j=1}^{l_1} \subset \mathbb{C}_+$ and $\{b_s\}_{s=1}^{l_2} \subset \mathbb{C}_+$ be those

zeros of functions $\psi_1(k)$ and $\psi_2(k)$ respectively. Consider functions $\mu_1(k)$ and $\mu_2(k)$ defined as

$$\mu_1(k) = \prod_{j=1}^r \left(\frac{1}{k - \alpha_j} \right) \cdot \prod_{j=1}^{l_1} \left(\frac{k - \bar{a}_j}{k - a_j} \right) \psi_1(k), k \in \mathbb{C}_+, \quad (34)$$

$$\mu_2(k) = \prod_{j=1}^r \left(\frac{1}{k - \alpha_j} \right) \cdot \prod_{j=1}^{l_2} \left(\frac{k - \bar{b}_j}{k - b_j} \right) \psi_2(k), k \in \mathbb{C}_+. \quad (35)$$

Then

$$\mu_1(k) \neq 0, \mu_2(k) \neq 0 \text{ for } k \in \mathbb{C}_+ \cup \mathbb{R}. \quad (36)$$

Furthermore, since

$$\left| \frac{k - \bar{a}}{k - a} \right| = 1, \forall k \in \mathbb{R}, \forall a \in \mathbb{C},$$

then (30), (34) and (35) imply that

$$|\mu_1(k)| = |\mu_2(k)|, \forall k \in \mathbb{R}. \quad (37)$$

To apply Lemma 2.4, we now should show that the asymptotic behavior of functions $\psi_1(k)$ and $\psi_2(k)$ for $|k| \rightarrow \infty, k \in \mathbb{C}_+$ is the same. It is natural to use formula (22), which would be similar to [22]. However, since functions $\psi_1(k)$ and $\psi_2(k)$ supposedly correspond to two different coefficients $c_1(x)$ and $c_2(x)$, then they generate different pairs of functions $\tau_1(x, y), A_1(x, y)$ and $\tau_2(x, y), A_2(x, y)$. This, in turn generates two different asymptotic behaviors in (22).

Nevertheless, we still can use (22). Indeed, since $y \in S, x \in B_\omega(y)$ and also $B_\omega(y) \cap \bar{\Omega} = \emptyset$, then $c(x) = 1$ in $B_\omega(y)$. Hence, $\tau(x, y) = |x - y|$ for $x \in B_\omega(y)$. Furthermore, by (14) $J(x, y) = 1$ for $x \in B_\omega(y)$. Hence, by (18)

$$A(x, y) = \frac{1}{4\pi |x - y|}, x \in B_\omega(y).$$

Hence, (22) implies that

$$u(x, y, k) = \frac{\exp(ik|x - y|)}{4\pi |x - y|} \left(1 + O\left(\frac{1}{k}\right) \right), |k| \rightarrow \infty, k \in \mathbb{C}_m, y \in S, x \in B_\omega(y).$$

Hence,

$$\psi_j(k) = \frac{\exp(ik|x - y|)}{4\pi |x - y|} \left(1 + O\left(\frac{1}{k}\right) \right), |k| \rightarrow \infty, k \in \mathbb{C}_m, j = 1, 2. \quad (38)$$

Hence, using (34), (35) and (38), we obtain

$$\mu_j(k) = \frac{1}{k^r} \frac{\exp(ik|x-y|)}{4\pi|x-y|} \left(1 + O\left(\frac{1}{k}\right)\right), |k| \rightarrow \infty, k \in \mathbb{C}_m, j = 1, 2. \quad (39)$$

Hence, (36), (37) and (39) imply that we can apply Lemma 2.4 to functions $\mu_1(k)$ and $\mu_2(k)$. We obtain then

$$\mu_1(k) = \mu_2(k), k \in \mathbb{C}_+ \cup \mathbb{R}. \quad (40)$$

Using (34), (35) and (40), we obtain

$$\prod_{j=1}^{l_1} \left(\frac{k - \bar{a}_j}{k - a_j}\right) \psi_1(k) = \prod_{j=1}^{l_2} \left(\frac{k - \bar{b}_j}{k - b_j}\right) \psi_2(k), k \in \mathbb{R}.$$

Or, equivalently,

$$\prod_{j=1}^{l_2} \left(\frac{k - b_j}{k - \bar{b}_j}\right) \psi_1(k) = \prod_{j=1}^{l_1} \left(\frac{k - a_j}{k - \bar{a}_j}\right) \psi_2(k), k \in \mathbb{R}. \quad (41)$$

We now want to apply the operator \mathcal{F}^{-1} of the inverse Fourier transform (26) to both sides of (41). We rewrite (41) as

$$\psi_1(k) + \left(\prod_{j=1}^{l_2} \frac{k - b_j}{k - \bar{b}_j} - 1\right) \psi_1(k) = \psi_2(k) + \left(\prod_{j=1}^{l_1} \frac{k - a_j}{k - \bar{a}_j} - 1\right) \psi_2(k), k \in \mathbb{R}. \quad (42)$$

Consider $\mathcal{F}^{-1}(\psi_1)$ and $\mathcal{F}^{-1}(\psi_2)$. By (20) and (21)

$$\mathcal{F}^{-1}(\psi_1) = v_1(x, y, t), \mathcal{F}^{-1}(\psi_2) = v_2(x, y, t), \quad (43)$$

where $v_1(x, y, t)$ and $v_2(x, y, t)$ are solutions of the problem (15) and (16) with two different functions $c_1(x)$ and $c_2(x)$ which correspond to functions $u_1(x, y, k)$ and $u_2(x, y, k)$ respectively. Let $d = \text{dist}(y, \partial\Omega)$. Then $d > \omega$. On the other hand, since $x \in B_\omega(y)$, then $|x - y| < \omega < d$. Let $t \in (0, d)$. Since $c(x') = 1, \forall x' \in B_\omega(y)$, then for $j = 1, 2$

$$\partial_t^2 v_j = \Delta v_j + \delta(x - y, t), x \in B_\omega(y), t \in (0, d), \quad (44)$$

$$v_j|_{t < 0} \equiv 0. \quad (45)$$

Hence, it follows from the method of energy estimates that for $j = 1, 2$

$$v_j(x, y, t) = \frac{\delta(t - |x - y|)}{4\pi|x - y|}, x \in B_\omega(y), t \in (0, d). \quad (46)$$

Hence, using (43) and (46), we obtain

$$\mathcal{F}^{-1}(\psi_1) = \mathcal{F}^{-1}(\psi_2) = \frac{\delta(t - |x - y|)}{4\pi|x - y|}, t \in (0, d). \quad (47)$$

Consider now functions $w_1(k), w_2(k)$ defined as

$$w_1(k) = \prod_{j=1}^{l_1} \frac{k - a_j}{k - \bar{a}_j} - 1,$$

$$w_2(k) = \prod_{j=1}^{l_2} \frac{k - b_j}{k - \bar{b}_j} - 1.$$

They can be rewritten as

$$w_1(k) = P_1(k) \prod_{j=1}^{l_1} \frac{1}{k - \bar{a}_j},$$

$$w_2(k) = P_2(k) \prod_{j=1}^{l_2} \frac{1}{k - \bar{b}_j},$$

where $P_1(k)$ is the polynomial of the degree less than l_1 and $P_2(k)$ is the polynomial of the degree less than l_2 . Using the partial fraction expansion, we obtain

$$w_1(k) = \sum_{j_1=1}^{l'_1} \frac{Y_{j_1}}{(k - \bar{a}_{j_1})^{p_{j_1}}},$$

$$w_2(k) = \sum_{j_2=1}^{l'_2} \frac{Z_{j_2}}{(k - \bar{b}_{j_2})^{q_{j_2}}},$$

where $l'_1 \leq l_1, l'_2 \leq l_2$, the sets $\{\bar{a}_{j_1}\}_{j_1=1}^{l'_1} \subseteq \{\bar{a}_j\}_{j=1}^{l_1}, \{\bar{b}_{j_2}\}_{j_2=1}^{l'_2} \subseteq \{\bar{b}_j\}_{j=1}^{l_2}$ and Y_{j_1}, Z_{j_2} are some complex numbers.

We now apply the operator \mathcal{F}^{-1} to functions $w_1(k), w_2(k)$. By (25)-(27)

$$\mathcal{F}^{-1}\left(\frac{Y_{j_1}}{(k - \bar{a}_{j_1})^{p_{j_1}}}\right) = H(t) \frac{(-1)^{p_{j_1}-1} i^{p_{j_1}} Y_{j_1}}{(p_{j_1} - 1)!} t^{p_{j_1}-1} \exp(-i\bar{a}_{j_1} t).$$

Hence,

$$\mathcal{F}^{-1}(w_1) = Q_1(t), \mathcal{F}^{-1}(w_2) = Q_2(t),$$

$$Q_1(t) = H(t) \sum_{j_1=1}^{l'_1} \frac{(-1)^{p_{j_1}-1} i^{p_{j_1}} Y_{j_1}}{(p_{j_1}-1)!} t^{p_{j_1}-1} \exp(-i\bar{a}_{j_1} t), \quad (48)$$

$$Q_2(t) = H(t) \sum_{j_2=1}^{l'_2} \frac{(-1)^{q_{j_2}-1} i^{q_{j_2}} Z_{j_2}}{(q_{j_2}-1)!} t^{p_{j_1}-1} \exp(-i\bar{b}_{j_2} t). \quad (49)$$

We are ready now to apply the operator \mathcal{F}^{-1} to both parts of (42). Using the convolution theorem for the Fourier transform, (43)-(45), (48) and (49), we obtain

$$v_1(x, y, t) + \int_0^t v_1(x, y, t - \tau) Q_1(\tau) d\tau = v_2(x, y, t) + \int_0^t v_2(x, y, t - \tau) Q_2(\tau) d\tau. \quad (50)$$

Let $t \in (0, d)$. Then, using (46), we obtain that (50) becomes

$$\int_0^t \delta(t - \tau - |x - y|) Q_1(\tau) d\tau = \int_0^t \delta(t - \tau - |x - y|) Q_2(\tau) d\tau, \forall t \in (0, d).$$

This is equivalent with

$$Q_1(t - |x - y|) = Q_2(t - |x - y|), \forall t \in (0, d). \quad (51)$$

By (48) and (49) both functions $Q_1(t), Q_2(t)$ are analytic functions of the real variable $t > 0$. Hence, since $d > |x - y|$, then (51) is valid for all $t > |x - y|$. Hence, denoting $\tilde{t} = t - |x - y|$, we obtain

$$Q_1(\tilde{t}) = Q_2(\tilde{t}), \forall \tilde{t} > 0. \quad (52)$$

Finally applying Lemma 2.6 to (52), we obtain that $l_1 = l_2 := l$ and sets $\{a_1, a_2, \dots, a_l\} = \{b_1, b_2, \dots, b_l\}$. Thus, by (41) $\psi_1(k) = \psi_2(k)$ for $k \in \mathbb{R}$. This and (29) lead to

$$u_1(x, y, k) = u_2(x, y, k), k \in \mathbb{R}. \quad (53)$$

So, (53) finalizes the proof of the first part of Theorem 1.

We now prove that the coefficient $c(x)$ is determined uniquely. The equality (53) is valid for that fixed pair $y \in S, x \in B_\omega(y)$. Since x is an arbitrary point of $B_\omega(y)$, then (53) holds for all $x \in B_\omega(y)$. We now return to the original equation (8). We have proven that for each source position $y \in S$ the function $u(x, y, k)$ is determined uniquely for points $x \in B_\omega(y)$

and for all $k \in \mathbb{R}$. Fix a point $y \in S$. Since by (1) and (2) $c(x) = 1$ outside of the domain Ω and since $B_\omega(y) \cap \overline{\Omega} = \emptyset$, then the well known theorem about the uniqueness of the continuation of the solution of an arbitrary elliptic equation of the second order (see, e.g. Chapter 4 of [28]) implies that the function $u(x, y, k)$ is determined uniquely for all $x \notin \overline{\Omega}$. Next, since $y \in S$ is an arbitrary point, then the function $u(x, y, k)$ is uniquely determined for all $y \in S$ and for all $x \notin \overline{\Omega}$. By (20) and (21) this means, in turn that the function $v(x, y, t)$ is uniquely determined for all $y \in S, x \notin \overline{\Omega}, t > 0$. This and (17) imply that the following function $s(x, y)$ is known

$$s(x, y) = \tau(x, y), \forall x, y \in S. \quad (54)$$

Recall that the function $\tau(x, y)$ satisfies the eikonal equation (5) with condition (6). The problem of the determination of the function $c(x)$ from the function $\tau(x, y)$ known for all $x, y \in S$ is called Inverse Kinematic Problem [28, 38, 39]. In the 3-D case uniqueness of this problem was proven in Theorem 3.4 of Chapter 3 of [39]. Therefore the uniqueness of the determination of the unknown coefficient $c(x)$ is established. \square

References

- [1] H. Ammari, Y.T. Chow and J. Zou, Phased and phaseless domain reconstruction in inverse scattering problem via scattering coefficients, *arxiv*: 1510.03999, 2015.
- [2] G. Bao, P. Li and J. Lv, Numerical solution of an inverse diffraction grating problem from phaseless data, *J. Optical Society of America A*, 30, 293-299, 2013.
- [3] G. Bao and L. Zhang, Shape reconstruction of the multi-scale rough surface from multi-frequency phaseless data, *Inverse Problems*, 32, 085002, 2016.
- [4] G. Bao, P. Li, J. Lin and F. Triki, Inverse scattering problems with multi-frequencies, *Inverse Problems*, 31, 093001, 2015.
- [5] W. Ballmann, *Lecture on Spaces of Nonpositive Curvature* (DMV-Seminar, Band 25), Birkhäuser Verlag, Basel, 1995.
- [6] L. Beilina and M.V. Klibanov, *Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems*, Springer, New York, 2012.

- [7] L. Beilina, Energy estimates and numerical verification of the stabilized domain decomposition finite element/finite difference approach for the Maxwell's system in time domain, *Central European Journal of Mathematics*, 11, 702–733, 2013.
- [8] K. Chadan and P.C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, Springer-Verlag, New York, 1977.
- [9] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, New York, 1992.
- [10] A.V. Darahanau, A.Y. Nikulin, A. Souvorov, Y. Nishino, B.C. Muddle and T. Ishikawa, Nano-resolution profiling of micro-structures using quantitative X-ray phase retrieval from Fraunhofer diffraction data, *Physics Letters A*, 335, 494–498, 2005.
- [11] M. Dierolf, O. Bank, S. Kynde, P. Thibault, I. Johnson, A. Menzel, K. Jefimovs, C. David, O. Marti and F. Pfeiffer, Ptychography & lensless X-ray imaging, *Europhysics News*, 39, 22-24, 2008.
- [12] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York, 1984.
- [13] G. Hu, J. Li, H. Liu and H. Sun, Inverse elastic scattering for multiscale rigid bodies with a single far-field pattern, *SIAM J. Imaging Sciences*, 7, 1799-1825, 2014.
- [14] V. Isakov, *Inverse Problems for Partial Differential Equations*, Second Edition, Springer, New York, 2006.
- [15] O. Ivanyshyn, R. Kress and P. Serranho, Huygens' principle and iterative methods in inverse obstacle scattering, *Advances in Computational Mathematics*, 33, 413-429, 2010.
- [16] O. Ivanyshyn and R. Kress, Inverse scattering for surface impedance from phase-less far field data, *J. Computational Physics*, 230, 3443-3452, 2011.
- [17] M.V. Klibanov and P.E. Sacks, Phaseless inverse scattering and the phase problem in optics, *J. Math. Physics*, 33, 3813-3821, 1992.
- [18] M.V. Klibanov, P.E. Sacks and A.V. Tikhonravov, The phase retrieval problem. Topical Review. *Inverse Problems*, 11, 1-28, 1995.

- [19] M.V. Klibanov, Phaseless inverse scattering problems in three dimensions, *SIAM J. Appl. Math.*, 74, 392-410, 2014.
- [20] M.V. Klibanov, On the first solution of a long standing problem: Uniqueness of the phaseless quantum inverse scattering problem in 3-d, *Applied Mathematics Letters*, 37, 82-85, 2014.
- [21] M.V. Klibanov, Uniqueness of two phaseless non-overdetermined inverse acoustics problems in 3-d, *Applicable Analysis*, 93, 1135-1149, 2014.
- [22] M.V. Klibanov and V.G. Romanov, Reconstruction procedures for two inverse scattering problems without the phase information, *SIAM J. Appl. Math.*, 76, 178-196, 2016.
- [23] M.V. Klibanov and V.G. Romanov, Two reconstruction procedures for a 3-D phaseless inverse scattering problem for the generalized Helmholtz equation, *Inverse Problems*, 32, 015005, 2016.
- [24] M.V. Klibanov and V.G. Romanov, The first solution of a long standing problem: Reconstruction formula for a 3-d phaseless inverse scattering problem for the Schrödinger equation, *J. Inverse and Ill-Posed Problems*, 23, 415-426, 2015.
- [25] M.V. Klibanov and V.G. Romanov, Explicit formula for the solution of the phaseless inverse scattering problem of imaging of nano structures, *J. Inverse and Ill-Posed Problems*, 23, 187-193, 2015.
- [26] M.V. Klibanov, L.H. Nguyen and K. Pan, Nanostructures imaging via numerical solution of a 3-D inverse scattering problem without the phase information, *arxiv*: 1510.00659, 2015.
- [27] O.A. Ladyzhenskaya, *Boundary Value Problems of Mathematical Physics*, Springer, New York, 1985.
- [28] M.M. Lavrentiev, V.G. Romanov and S.P. Shishatskii, *Ill-Posed Problems of Mathematical Physics and Analysis*, AMS, Providence, RI, 1986.
- [29] J. Li, H. Liu, Z. Shang and H. Sun, Two single-shot methods for locating multiple electromagnetic scatterers, *SIAM J. Appl. Math.*, 73, 1721-1746, 2013.

- [30] R.G. Newton, *Inverse Schrödinger Scattering in Three Dimensions*, Springer, New York, 1989.
- [31] R.G. Novikov, A multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$, *Funct. Anal. Appl.*, 22, 263–272, 1988.
- [32] R.G. Novikov, The inverse scattering problem on a fixed energy level for the two-dimensional Schrödinger operator, *J. Functional Analysis*, 103, 409–463, 1992.
- [33] R.G. Novikov, Explicit formulas and global uniqueness for phaseless inverse scattering in multidimensions, *J. Geometrical Analysis*, 26, 346–359, 2016.
- [34] R.G. Novikov, Formulas for phase recovering from phaseless scattering data at fixed frequency, *Bulletin des Sciences Mathématiques*, 139, 923–936, 2015.
- [35] T. C. Petersena, V.J. Keastb and D. M. Paganinc, Quantitative TEM-based phase retrieval of MgO nano-cubes using the transport of intensity equation, *Ultramicroscopy*, 108, 805–815, 2008.
- [36] R. Phillips and R. Milo, A feeling for numbers in biology, *Proc. Natl. Acad. Sci. USA*, 106, 21465–71, 2009.
- [37] <http://kirschner.med.harvard.edu/files/bionumbers/fundamentalBioNumbersHandout.pdf>
- [38] V.G. Romanov, *Integral Geometry and Inverse Problems for Hyperbolic Equations*, Springer - Verlag, Berlin, 1974.
- [39] V.G. Romanov, *Inverse Problems of Mathematical Physics*, VNU Science Press, Utrecht, 1987.
- [40] V.G. Romanov, Inverse problems for differential equations with memory, *Eurasian J. of Mathematical and Computer Applications*, 2, issue 4, 51–80, 2014.
- [41] A. Ruhlandt, M. Krenkel, M. Bartels, and T. Salditt, Three-dimensional phase retrieval in propagation-based phase-contrast imaging, *Physical Review A*, 89, 033847, 2014.
- [42] N. T. Thành, L. Beilina, M. V. Klibanov and M. A. Fiddy, Reconstruction of the refractive index from experimental backscattering data

- using a globally convergent inverse method, *SIAM Journal on Scientific Computing*, 36, B273–B293, 2014.
- [43] N. T. Thành, L. Beilina, M. V. Klibanov and M. A. Fiddy, Imaging of buried objects from experimental backscattering time dependent measurements using a globally convergent inverse algorithm, *SIAM J. Imaging Sciences*, 8, 757-786, 2015.
- [44] B.R. Vainberg, Principles of radiation, limiting absorption and limiting amplitude in the general theory of partial differential equations, *Russian Math. Surveys*, 21, 115-193, 1966.
- [45] B.R. Vainberg, *Asymptotic Methods in Equations of Mathematical Physics*, Gordon and Breach Science Publishers, New York, 1989.